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Existence of positive solutions for a system of generalized Lidstone problems[☆]

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ABSTRACT

In this paper we study the existence of positive solutions for the system of generalized Lidstone problems for ordinary differential equations

$$\begin{cases}(-1)^m u^{(2m)} = f_1(t, u, -u'', \ldots, (-1)^{m-1}u^{(2m-2)}, v, -v'', \ldots, (-1)^{n-1}v^{(2n-2)}),\\ (-1)^n v^{(2n)} = f_2(t, u, -u'', \ldots, (-1)^{m-1}u^{(2m-2)}, v, -v'', \ldots, (-1)^{n-1}v^{(2n-2)}),\\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = \alpha_1 u^{(2i)}(1) + \beta_1 u^{(2i+1)}(1) = 0\\ \quad (i = 0, 1, \ldots, m-1),\\ \alpha_0 v^{(2j)}(0) - \beta_0 v^{(2j+1)}(0) = \alpha_1 v^{(2j)}(1) + \beta_1 v^{(2j+1)}(1) = 0\\ \quad (j = 0, 1, \ldots, n-1).\end{cases}$$

Here $f_1, f_2 \in C([0, 1] \times \mathbb{R}_+^{m+n}, \mathbb{R}_+)(\mathbb{R}_+ := [0, +\infty))$. Our hypotheses imposed on the nonlinearities f_1 and f_2 are formulated in terms of nonnegative matrices. We use fixed point index theory to establish our main results based on a priori estimates of positive solutions.

1. Introduction

In this paper, we study the existence of positive solutions to the system of generalized Lidstone problems for ordinary differential equations

$$\begin{cases}(-1)^m u^{(2m)} = f_1(t, u, -u'', \ldots, (-1)^{m-1}u^{(2m-2)}, v, -v'', \ldots, (-1)^{n-1}v^{(2n-2)}),\\ (-1)^n v^{(2n)} = f_2(t, u, -u'', \ldots, (-1)^{m-1}u^{(2m-2)}, v, -v'', \ldots, (-1)^{n-1}v^{(2n-2)}),\\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = \alpha_1 u^{(2i)}(1) + \beta_1 u^{(2i+1)}(1) = 0 \quad (i = 0, 1, \ldots, m-1),\\ \alpha_0 v^{(2j)}(0) - \beta_0 v^{(2j+1)}(0) = \alpha_1 v^{(2j)}(1) + \beta_1 v^{(2j+1)}(1) = 0 \quad (j = 0, 1, \ldots, n-1).\end{cases} \tag{1.1}$$

Here $m \geq 1, n \geq 1, \alpha_k, \beta_k \in \mathbb{R}_+$ $(k = 0, 1, \mathbb{R}_+ := [0, +\infty))$, $\alpha_0\alpha_1 + \alpha_0\beta_1 + \alpha_1\beta_0 > 0, f_1, f_2 \in C([0, 1] \times \mathbb{R}_+^{m+n}, \mathbb{R}_+)$. Note by a positive solution we mean a pair of functions $(u, v) \in C^{2m}[0, 1] \times C^{2n}[0, 1]$ that solve (1.1) and satisfy $u(t) \geq 0$ and $v(t) \geq 0$ for $t \in [0, 1]$, with at least one of them positive on $(0, 1)$.

In recent years, the so-called Lidstone problem

$$\begin{cases}(-1)^n u^{(2n)} = f(t, u, -u'', \ldots, (-1)^{n-1}u^{(2n-2)}),\\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, \ldots, n-1,\end{cases} \tag{1.2}$$

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has been extensively studied, see [1–18] and references cited therein. In [19], the author studied the existence and uniqueness of positive solutions for the generalized Lidstone problem

$$\begin{cases}(-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1}u^{(2n-2)}),\\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = 0 \quad (i = 0, 1, \dots, n-1),\\ \alpha_1 u^{(2i)}(1) + \beta_1 u^{(2i+1)}(1) = 0 \quad (i = 0, 1, \dots, n-1),\end{cases} \tag{1.3}$$

where $f \in C([0, 1] \times \mathbb{R}_+^n, \mathbb{R}_+)$, and α_i, β_i $(i = 0, 1)$ are as in (1.1). The main results obtained in [19] are given in terms of the spectral radii of some linear integral operators associated with the nonlinearity f and the boundary conditions in (1.3), thereby extending the corresponding sharp results for second order boundary value problems due to Liu and Li in 1996 (see [20]).

Many authors have studied the existence of positive solutions for the system of Dirichlet problems

$$\begin{cases}-u'' = f(t, u, v),\\ -v'' = g(t, u, v),\\ u(0) = u(1) = 0,\\ v(0) = v(1) = 0,\end{cases} \tag{1.4}$$

where $f, g \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. For more details, see, for instance, [21–24]. Recently, in [25], using fixed point theory, Infante et al. considered the existence and multiplicity of positive solutions for a system of perturbed Hammerstein integral equations, with the main results applied to a wide class of boundary value problems. To the best of our knowledge, however, the generic problem (1.1) has not been treated in the literature.

It should be remarked that the orders $2m$ and $2n$ in (1.1) may be different. Such systems of boundary value problems with different orders can be encountered in applied sciences, see [26]. For example, the system of boundary value problems

$$\begin{cases}u^{(4)} = f(t, v),\\ -v'' = g(t, u),\\ u(0) = u(1) = u''(0) = u''(1) = 0,\\ v(0) = v(1) = 0,\end{cases} \tag{1.5}$$

studied in [27], arises from the problem of suspension bridge equations. Note that (1.5) is special case of (1.1) (with $m = 2$ and $n = 1$). In [28], Kang et al. also discussed a system of boundary value problems with possibly different orders.

In order to overcome the difficulty of (1.1) resulting from higher order derivatives, as in [19], we use the method of order reduction to transform (1.1) into a system of boundary value problems for second-order integro-ordinary differential equations. It is of interest to note that our hypotheses imposed on the nonlinearities f_1 and f_2 are formulated in terms of nonnegative matrices. We use fixed point theory to establish our main results based on a priori estimates achieved by utilizing nonnegative matrices. We refer the reader to [25,29] and references therein for interesting applications of fixed point index theory, combined with nonnegative matrices, in the existence of positive solutions for nonlinear Hammerstein integral equations and boundary value problems for nonlinear ordinary differential equations.

This paper is organized as follows. Section 2 contains some preliminary results, including some basic facts recalled from [19]. The main results are stated and proved in Section 3. Finally, in Section 4, we provide two examples to illustrate our main results.

2. Preliminaries

Let

$$E := C([0, 1], \mathbb{R}), \qquad \|u\| := \max_{t\in[0,1]} |u(t)|, \qquad P := \{u \in E : u(t) \geq 0, \forall t \in [0, 1]\}.$$

Clearly, $(E, \|\cdot\|)$ is a real Banach space and P is a solid cone (see [30, p. 193]) on E.

Throughout this paper we make assumptions on α_i, β_i $(i = 0, 1)$ as follows.

(H1) $\alpha_i \geq 0$, $\beta_i \geq 0$, $\Delta := \alpha_0\alpha_1 + \alpha_0\beta_1 + \alpha_1\beta_0 > 0$.

For the convenience of the reader, we recall some basic facts from [19].

It is easy to see that

$$G_1(t, s) := \frac{1}{\Delta}\begin{cases}(\beta_0 + \alpha_0 t)(\alpha_1 + \beta_1 - \alpha_1 s), & 0 \leq t \leq s \leq 1,\\ (\beta_0 + \alpha_0 s)(\alpha_1 + \beta_1 - \alpha_1 t), & 0 \leq s \leq t \leq 1,\end{cases} \tag{2.1}$$

is the Green's function for the linear boundary value problem

$$\begin{cases}-u'' = 0,\\ \alpha_0 u(0) - \beta_0 u'(0) = 0,\\ \alpha_1 u(1) + \beta_1 u'(1) = 0.\end{cases}$$

With G_1 given by (2.1), we define the completely continuous linear operators $L : E \to E$ by

$$(Lu)(t) := \int_0^1 G_1(t,s)u(s)\mathrm{d}s. \tag{2.2}$$

Then L is also a positive operator, i.e. $L(P) \subset P$. Let

$$G_i(t,s) := \int_0^1 G_1(t,\tau)G_{i-1}(\tau,s)\mathrm{d}\tau \quad (i = 2, \ldots). \tag{2.3}$$

Apparently we have

$$(L^i u)(t) = \int_0^1 G_i(t,s)u(s)\mathrm{d}s, \quad u \in E$$

for $i = 1, 2, \ldots$. Moreover, L has the spectral radius $r(L) > 0$, whence $r(L^i) = r^i(L) > 0$ for each positive integer i. Let $\lambda_1 > 0$ be the first eigenvalue of and $\varphi \in C^2[0,1] \cap P$ the associated eigenfunction of

$$\begin{cases} -u'' = \lambda u, \\ \alpha_0 u(0) - \beta_0 u'(0) = 0, \\ \alpha_1 u(1) + \beta_1 u'(1) = 0, \end{cases}$$

with $\int_0^1 \varphi(t)\mathrm{d}t = 1$, which can be written in the form

$$\varphi(t) = \lambda_1 \int_0^1 G_1(t,s)\varphi(s)\mathrm{d}s = \lambda_1 (L\varphi)(t). \tag{2.4}$$

Therefore $\lambda_1 = 1/r(L)$. Moreover, the symmetry of $G_1(t,s)$ implies that

$$\varphi(s) = \lambda_1^i \int_0^1 G_i(t,s)\varphi(t)\mathrm{d}t = \lambda_1^i (L^i\varphi)(s) \tag{2.5}$$

for $i = 1, 2, \ldots$.

Lemma 1 (*See [31, Lemma 2]*). *Let G_1 be defined by (2.1) and*

$$h(t) := \frac{1}{N}\min\{\alpha_0 + \beta_0 t, \alpha_1 + \beta_1 - \alpha_1 t\}, \tag{2.6}$$

where $N := \max\{\alpha_0 + \beta_0, \alpha_1 + \beta_1\}$. Then

$$G_1(t,s) \geq h(t)G_1(\tau,s), \quad \forall t, s, \tau \in [0,1].$$

Let h be given by (2.6) and φ by (2.4). Put $\kappa := \int_0^1 h(t)\varphi(t)\mathrm{d}t > 0$ and

$$P_0 := \left\{ u \in P : \int_0^1 \varphi(t)u(t)\mathrm{d}t \geq \kappa \|u\| \right\}.$$

Clearly, P_0 is also a cone on E.

Lemma 2 (*See [19, Lemma 2]*). *If* (H1) *holds, then $L(P) \subset P_0$.*

Let $w(t) := (-1)^{n-1}u^{(2m-2)}(t), z(t) := (-1)^{n-1}v^{(2n-2)}(t)$. Then (1.1) is equivalent to the system of integro-ordinary differential equations

$$\begin{cases} -w'' = f_1(t, L^{m-1}w, \ldots, w, L^{n-1}z, \ldots, z), \\ -z'' = f_2(t, L^{m-1}w, \ldots, w, L^{n-1}z, \ldots, z), \\ \alpha_0 w(0) - \beta_0 w'(0) = \alpha_1 w(1) + \beta_1 w'(1) = 0, \\ \alpha_0 z(0) - \beta_0 z'(0) = \alpha_1 z(1) + \beta_1 z'(1) = 0 \end{cases} \tag{2.7}$$

which can be written in the form

$$\begin{cases} w(t) = \displaystyle\int_0^1 G_1(t,s)f_1(s, (L^{m-1}w)(s), \ldots, w(s), (L^{n-1}z)(s), \ldots, z(s))\mathrm{d}s, \\ z(t) = \displaystyle\int_0^1 G_1(t,s)f_2(s, (L^{m-1}w)(s), \ldots, w(s), (L^{n-1}z)(s), \ldots, z(s))\mathrm{d}s. \end{cases} \tag{2.8}$$

Define operators T_1, T_2 and T by

$$T_i(w,z)(t) = \int_0^1 G_1(t,s) f_i(s, (L^{m-1}w)(s), \dots, w(s), (L^{n-1}z)(s), \dots, z(s))\mathrm{d}s,$$
$$T(w,z)(t) = (T_1(w,z)(t), T_2(w,z)(t)).$$

Then $T_i : P \times P \to P$ $(i = 1, 2)$ and $T : P \times P \to P \times P$ are completely continuous operators.

Lemma 3 (*See [32,33]*). *Let E be a real Banach space and P a cone in E. Suppose that $\Omega \subset E$ is a bounded open set and that $T : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $w_0 \in P \setminus \{0\}$ such that $w - Tw \neq \theta w_0$ for all $\theta \geq 0$, $w \in \partial\Omega \cap P$, then $i(T, \Omega \cap P, P) = 0$, where i indicates the fixed point index on P.*

Lemma 4 (*See [32,33]*). *Let E be a real Banach space and P a cone in E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $T : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If $w - \theta Tw \neq 0$ for all $\theta \in [0, 1]$, $w \in \partial\Omega \cap P$, then $i(T, \Omega \cap P, P) = 1$.*

3. Main results

Definition 1. A real matrix A is said to be nonnegative if all elements of A are nonnegative.

Remark 1. A real nonnegative matrix, viewed as a linear operator acting on a suitable Euclidean space, is increasing in the sense of components. Hence the term of order preserving matrix in [25,29].

For the reason of notational brevity, we denote by $x := (x_1, x_2, \dots, x_m) \in \mathbb{R}_+^m$ and $y := (y_1, y_2, \dots, y_n) \in \mathbb{R}_+^n$. We now list our hypotheses on f_1 and f_2.

(H2) $f_1, f_2 \in C([0, 1] \times \mathbb{R}_+^{m+n}, \mathbb{R}_+)$.

(H3) There are two nonnegative matrices $(a_{ij})_{2\times m}$, $(b_{ik})_{2\times n}$, and a real number $c > 0$ such that

$$f_i(t, x, y) \geq \sum_{j=1}^m a_{ij}x_j + \sum_{k=1}^n b_{ik}y_k - c \quad (i = 1, 2)$$

for all $(t, x, y) \in [0, 1] \times \mathbb{R}_+^{m+n}$ and the matrix $A_1 := \begin{pmatrix} a_1 - 1 & b_1 \\ a_2 & b_2 - 1 \end{pmatrix}$ is invertible with A_1^{-1} nonnegative, where

$$a_i := \sum_{j=1}^m a_{ij}(r(L))^{m-j+1}, \qquad b_i := \sum_{k=1}^n b_{ik}(r(L))^{n-k+1} \quad (i = 1, 2).$$

(H4) There are two nonnegative matrices $(c_{ij})_{2\times m}$, $(d_{ik})_{2\times n}$, and a real number $r > 0$ such that

$$f_i(t, x, y) \leq \sum_{j=1}^m c_{ij}x_j + \sum_{k=1}^n d_{ik}y_k, \quad (i = 1, 2)$$

for all $(t, x, y) \in [0, 1] \times ([0, r])^{m+n}$ and the matrix $A_2 := \begin{pmatrix} 1 - c_1 & -d_1 \\ -c_2 & 1 - d_2 \end{pmatrix}$ is invertible with A_2^{-1} nonnegative, where

$$c_i := \sum_{j=1}^m c_{ij}(r(L))^{m-j+1}, \qquad d_i := \sum_{k=1}^n d_{ik}(r(L))^{n-k+1} \quad (i = 1, 2).$$

(H5) There are two nonnegative matrices $(l_{ij})_{2\times m}$, $(m_{ik})_{2\times n}$, and a real number $c > 0$ such that

$$f_i(t, x, y) \leq \sum_{j=1}^m l_{ij}x_j + \sum_{k=1}^n m_{ik}y_k + c, \quad (i = 1, 2)$$

for all $(t, x, y) \in [0, 1] \times \mathbb{R}_+^{m+n}$ and the matrix $A_3 := \begin{pmatrix} 1 - l_1 & -m_1 \\ -l_2 & 1 - m_2 \end{pmatrix}$ is invertible with A_3^{-1} nonnegative, where

$$l_i := \sum_{j=1}^m l_{ij}(r(L))^{m-j+1}, \qquad m_i := \sum_{k=1}^n m_{ik}(r(L))^{n-k+1} \quad (i = 1, 2).$$

(H6) There are two nonnegative matrices $(p_{ij})_{2\times m}$, $(q_{ik})_{2\times n}$, and a real number r such that

$$f_i(t, x, y) \geq \sum_{j=1}^m p_{ij}x_j + \sum_{k=1}^n q_{ik}y_k, \quad (i = 1, 2)$$

for all $(t, x, y) \in [0, 1] \times ([0, r])^{m+n}$ and the matrix $A_4 := \begin{pmatrix} p_1 - 1 & q_1 \\ p_2 & q_2 - 1 \end{pmatrix}$ is invertible with A_4^{-1} nonnegative, where

$$p_i := \sum_{j=1}^{m} p_{ij}(r(L))^{m-j+1}, \qquad q_i := \sum_{k=1}^{n} q_{ik}(r(L))^{n-k+1} \quad (i = 1, 2).$$

Remark 2. Let m_{ij} $(i, j = 1, 2)$ be four nonnegative constants. Then it is easy to see that the matrix $A := \begin{pmatrix} m_{11} - 1 & m_{12} \\ m_{21} & m_{22} - 1 \end{pmatrix}$ is invertible with A^{-1} nonnegative if and only if one of the following two conditions is satisfied:

(1) $m_{11} > 1, m_{22} > 1, m_{12} = m_{21} = 0$.
(2) $m_{11} \leq 1, m_{22} \leq 1, \det A = (1 - m_{11})(1 - m_{22}) - m_{12}m_{21} < 0$.

Remark 3. Let m_{ij} $(i, j = 1, 2)$ be four nonnegative constants. Then it is easy to see that the matrix $B := \begin{pmatrix} 1 - m_{11} & -m_{12} \\ -m_{21} & 1 - m_{22} \end{pmatrix}$ is invertible with B^{-1} nonnegative if and only if $m_{11} < 1, m_{22} < 1, \det B = (1 - m_{11})(1 - m_{22}) - m_{12}m_{21} > 0$.

Theorem 1. *If* (H1)–(H4) *hold, then* (1.1) *has at least one positive solution.*

Proof. By (H3), we have for every $(w, z) \in P \times P$ and $i = 1, 2$

$$\begin{aligned} T_i(w, z)(t) &\geq \sum_{\substack{1 \leq j \leq m \\ 1 \leq k \leq n}} \int_0^1 G_1(t, s)(a_{ij}(L^{m-j}w)(s) + b_{ik}(L^{n-k}z)(s))\mathrm{d}s - c\int_0^1 G_1(t, s)\mathrm{d}s \\ &= \sum_{\substack{1 \leq j \leq m \\ 1 \leq k \leq n}} \int_0^1 (G_{m-j+1}(t, s)a_{ij}w(s) + G_{n-k+1}b_{ik}z(s))\mathrm{d}s - c\int_0^1 G_1(t, s)\mathrm{d}s. \end{aligned} \tag{3.1}$$

Let

$$\mathscr{M}_1 := \{(w, z) \in P \times P : (w, z) = T(w, z) + \lambda(\varphi, \varphi), \lambda \geq 0\}$$

where $\varphi \in P \setminus \{0\}$ is given in (2.4). We want to show that $\mathscr{M}_1$ is bounded. Indeed, if $(w, z) \in \mathscr{M}_1$, then Lemma 2 implies that $w \in P_0$ and $z \in P_0$. Consequently, we have

$$\|w\| \leq \frac{1}{\kappa}\int_0^1 w(t)\varphi(t)\mathrm{d}t, \qquad \|z\| \leq \frac{1}{\kappa}\int_0^1 z(t)\varphi(t)\mathrm{d}t \tag{3.2}$$

for every $(w, z) \in \mathscr{M}_1$. In addition, by definition, the inequalities

$$w \geq T_1(w, z), \qquad z \geq T_2(w, z)$$

hold for every $(w, z) \in \mathscr{M}_1$. Combining this with (3.1) gives

$$w(t) \geq \sum_{\substack{1 \leq j \leq m \\ 1 \leq k \leq n}} \int_0^1 (G_{m-j+1}(t, s)a_{1j}w(s) + G_{n-k+1}b_{1k}z(s))\mathrm{d}s - c\int_0^1 G_1(t, s)\mathrm{d}s$$

and

$$z(t) \geq \sum_{\substack{1 \leq j \leq m \\ 1 \leq k \leq n}} \int_0^1 (G_{m-j+1}(t, s)a_{2j}w(s) + G_{n-k+1}b_{2k}z(s))\mathrm{d}s - c\int_0^1 G_1(t, s)\mathrm{d}s$$

if $(w, z) \in \mathscr{M}_1$. Multiply $\varphi(t)$ on both sides of the last two inequalities and integrate over $[0, 1]$ and use (2.5) to obtain

$$\int_0^1 w(t)\varphi(t)\mathrm{d}t \geq a_1\int_0^1 w(t)\varphi(t)\mathrm{d}t + b_1\int_0^1 z(t)\varphi(t)\mathrm{d}t - cr(L)$$

and

$$\int_0^1 z(t)\varphi(t)\mathrm{d}t \geq a_2\int_0^1 w(t)\varphi(t)\mathrm{d}t + b_2\int_0^1 z(t)\varphi(t)\mathrm{d}t - cr(L)$$

for every $(w, z) \in \mathscr{M}_1$. The last two inequalities can be written as

$$\begin{pmatrix} a_1 - 1 & b_1 \\ a_2 & b_2 - 1 \end{pmatrix}\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} = A_1\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} \leq \begin{pmatrix} cr(L) \\ cr(L) \end{pmatrix}.$$

Now (H3) implies

$$\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} \le A_1^{-1}\begin{pmatrix} cr(L) \\ cr(L) \end{pmatrix}.$$

Consequently, there exists a constant $M > 0$ such that

$$\int_0^1 w(t)\varphi(t)\mathrm{d}t \le M, \qquad \int_0^1 z(t)\varphi(t)\mathrm{d}t \le M, \quad \forall (w,z) \in \mathscr{M}_1.$$

Recalling (3.2), we obtain

$$\|w\| \le \frac{M}{\kappa}, \qquad \|z\| \le \frac{M}{\kappa}, \quad \forall (w,z) \in \mathscr{M}_1,$$

which proves the boundedness of $\mathscr{M}_1$. Taking $R > \sup\{\|(w,z)\| : (w,z) \in \mathscr{M}_1\}$, we have

$$(w,z) \neq T(w,z) + \lambda(\varphi,\varphi), \quad \forall (w,z) \in \partial\Omega_R \cap (P \times P), \lambda \ge 0.$$

Now Lemma 3 yields

$$i(T, \Omega_R \cap (P \times P), P \cap P) = 0. \tag{3.3}$$

By (H4), we have for every $(w,z) \in \overline{\Omega}_r \cap (P \times P)$ and $i = 1, 2$

$$\begin{aligned} T_i(w,z)(t) &\le \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 G_1(t,s)(c_{ij}(L^{m-j}w)(s) + d_{ik}(L^{n-k}z)(s))\mathrm{d}s \\ &= \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t,s)c_{ij}w(s) + G_{n-k+1}(t,s)d_{ik}z(s))\mathrm{d}s. \end{aligned} \tag{3.4}$$

Let

$$\mathscr{M}_2 := \{(w,z) \in \overline{\Omega}_r \cap (P \times P) : (w,z) = \lambda T(w,z), \lambda \in [0,1]\}.$$

Now we want to prove that $\mathscr{M}_2 = \{0\}$. Indeed, if $(w,z) \in \mathscr{M}_2$, then

$$w(t) \le T_1(w,z)(t) \le \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t,s)c_{1j}w(s) + G_{n-k+1}(t,s)d_{1k}z(s))\mathrm{d}s$$

and

$$z(t) \le T_2(w,z)(t) \le \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t,s)c_{2j}w(s) + G_{n-k+1}(t,s)d_{2k}z(s))\mathrm{d}s.$$

Multiply $\varphi(t)$ on both sides of the last two inequalities and integrate over [0, 1] and use (2.5) to obtain

$$\int_0^1 w(t)\varphi(t)\mathrm{d}t \le c_1\int_0^1 w(t)\varphi(t)\mathrm{d}t + d_1\int_0^1 z(t)\varphi(t)\mathrm{d}t$$

and

$$\int_0^1 z(t)\varphi(t)\mathrm{d}t \le c_2\int_0^1 w(t)\varphi(t)\mathrm{d}t + d_2\int_0^1 z(t)\varphi(t)\mathrm{d}t$$

for every $(w,z) \in \mathscr{M}_2$. That is,

$$\begin{pmatrix} 1-c_1 & -d_1 \\ -c_2 & 1-d_2 \end{pmatrix}\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} = A_2\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(H4) again implies

$$\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} \le A_2^{-1}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Consequently, $\int_0^1 w(t)\varphi(t)\mathrm{d}t = \int_0^1 z(t)\varphi(t)\mathrm{d}t = 0$ and whence $w \equiv 0$, $z \equiv 0$, as required. Thus we have

$$(w,z) \neq \lambda T(w,z), \quad \forall (w,z) \in \partial\Omega_r \cap (P\times P), \lambda \in [0,1].$$

Now Lemma 4 yields

$$i(T, \Omega_r \cap (P\times P), P\times P) = 1.$$

This, together with (3.3), implies

$$i(T, (\Omega_R \setminus \overline{\Omega}_r) \cap (P\times P), P\times P) = 0 - 1 = -1.$$

Consequently T has at least one fixed point on $(\Omega_R\setminus\overline{\Omega}_r)\cap(P\times P)$ and (2.8) has at least one positive solution (w,z). Thus (1.1) has at least one positive solution $(u,v) = (L^{m-1}w, L^{n-1}z)$. This completes the proof. □

Theorem 2. *If* (H1), (H2), (H5) *and* (H6) *hold, then* (1.1) *has at least one positive solution.*

Proof. By (H5), we have for every $(w,z) \in P\times P$ and $i = 1, 2$

$$\begin{aligned} T_i(w,z)(t) &\le \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 G_1(t,s)(l_{ij}(L^{m-j}w)(s) + m_{ik}(L^{n-k}z)(s))\mathrm{d}s + c\int_0^1 G_1(t,s)\mathrm{d}s \\ &= \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t,s)l_{ij}w(s) + G_{n-k+1}m_{ik}z(s))\mathrm{d}s + c\int_0^1 G_1(t,s)\mathrm{d}s. \end{aligned} \tag{3.5}$$

Let

$$\mathscr{M}_3 := \{(w,z) \in P\times P : (w,z) = \lambda T(w,z), \lambda \in [0,1]\}.$$

We now assert that $\mathscr{M}_3$ is bounded. Indeed, Lemma 2 implies that $w \in P_0$ and $z \in P_0$ if $(w,z) \in \mathscr{M}_3$, that is, (3.2) holds for every $(w,z) \in \mathscr{M}_3$. Moreover, by the definition of $\mathscr{M}_3$, we have

$$w(t) \le \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t,s)l_{1j}w(s) + G_{n-k+1}m_{1k}z(s))\mathrm{d}s + c\int_0^1 G_1(t,s)\mathrm{d}s$$

and

$$z(t) \le \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t,s)l_{2j}w(s) + G_{n-k+1}m_{2k}z(s))\mathrm{d}s + c\int_0^1 G_1(t,s)\mathrm{d}s.$$

Multiply $\varphi(t)$ on both sides of the last two inequalities and integrate over $[0, 1]$ and use (2.5) to obtain

$$\int_0^1 w(t)\varphi(t)\mathrm{d}t \le l_1\int_0^1 w(t)\varphi(t)\mathrm{d}t + m_1\int_0^1 z(t)\varphi(t)\mathrm{d}t + cr(L)$$

and

$$\int_0^1 z(t)\varphi(t)\mathrm{d}t \le l_2\int_0^1 w(t)\varphi(t)\mathrm{d}t + m_2\int_0^1 z(t)\varphi(t)\mathrm{d}t + cr(L)$$

for every $(w,z) \in \mathscr{M}_3$. The last two inequalities can be written in the form

$$\begin{pmatrix} 1-l_1 & -m_1 \\ -l_2 & 1-m_2 \end{pmatrix}\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} = A_3\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} \le \begin{pmatrix} cr(L) \\ cr(L) \end{pmatrix}.$$

(H5) again implies

$$\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} \le A_3^{-1} \begin{pmatrix} cr(L) \\ cr(L) \end{pmatrix}.$$

Consequently, there exists $M > 0$ such that

$$\int_0^1 w(t)\varphi(t)\mathrm{d}t \le M, \qquad \int_0^1 z(t)\varphi(t)\mathrm{d}t \le M, \quad \forall (w, z) \in \mathscr{M}_3.$$

By virtue of (3.2), we have

$$\|w\| \le \frac{M}{\kappa}, \qquad \|z\| \le \frac{M}{\kappa}, \quad \forall (w, z) \in \mathscr{M}_3.$$

This proves the boundedness of $\mathscr{M}_3$. Taking $R > \sup\{\|(w, z)\| : (w, z) \in \mathscr{M}_3\}$, we have

$$(w, z) \ne \lambda T(w, z), \quad \forall (w, z) \in \partial\Omega_R \cap (P \times P), \lambda \in [0, 1].$$

Now Lemma 4 yields

$$i(T, \Omega_R \cap (P \times P), P \times P) = 1. \tag{3.6}$$

By (H6), we have for every $(w, z) \in \overline{\Omega}_r \cap (P \times P)$ and $i = 1, 2$

$$\begin{aligned} T_i(w, z)(t) &\ge \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 G_1(t, s)(p_{ij}(L^{m-j}w)(s) + q_{ik}(L^{n-k}z)(s))\mathrm{d}s \\ &= \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t, s)p_{ij}w(s) + G_{n-k+1}q_{ik}z(s))\mathrm{d}s. \end{aligned} \tag{3.7}$$

Let

$$\mathscr{M}_4 := \{(w, z) \in \overline{\Omega}_r \cap (P \times P) : (w, z) \ge T(w, z)\}.$$

We want to prove that $\mathscr{M}_4 \subset \{0\}$. Indeed, if $(w, z) \in \mathscr{M}_4$, then (3.7) implies

$$w(t) \ge \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t, s)p_{1j}w(s) + G_{n-k+1}q_{1k}z(s))\mathrm{d}s$$

and

$$z(t) \ge \sum_{\substack{1\le j\le m\\1\le k\le n}} \int_0^1 (G_{m-j+1}(t, s)p_{2j}w(s) + G_{n-k+1}q_{2k}z(s))\mathrm{d}s.$$

Multiply $\varphi(t)$ on both sides of the last two inequalities and integrate over $[0, 1]$ and use (2.5) to obtain

$$\int_0^1 w(t)\varphi(t)\mathrm{d}t \ge p_1 \int_0^1 w(t)\varphi(t)\mathrm{d}t + q_1 \int_0^1 z(t)\varphi(t)\mathrm{d}t$$

and

$$\int_0^1 z(t)\varphi(t)\mathrm{d}t \ge p_2 \int_0^1 w(t)\varphi(t)\mathrm{d}t + q_2 \int_0^1 z(t)\varphi(t)\mathrm{d}t,$$

which can be written in the form

$$\begin{pmatrix} p_1 - 1 & q_1 \\ p_2 & q_2 - 1 \end{pmatrix} \begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} = A_4 \begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(H6) now implies

$$\begin{pmatrix} \int_0^1 w(t)\varphi(t)\mathrm{d}t \\ \int_0^1 z(t)\varphi(t)\mathrm{d}t \end{pmatrix} \le A_4^{-1}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Therefore $\int_0^1 w(t)\varphi(t)\mathrm{d}t = \int_0^1 z(t)\varphi(t)\mathrm{d}t = 0$ and thus $w \equiv 0$ and $z \equiv 0$. This implies $\mathscr{M}_4 \subset \{0\}$. As a result of this, we obtain

$$(w,z) \ne T(w,z) + \lambda(\varphi,\varphi), \quad \forall (w,z) \in \partial\Omega_r \cap (P\times P), \lambda \ge 0.$$

By Lemma 3, we have

$$i(T, \Omega_r \cap (P\times P), P\times P) = 0.$$

Combining this with (3.6) gives

$$i(T, (\Omega_R \setminus \overline{\Omega}_r) \cap (P\times P), P\times P) = 1 - 0 = 1.$$

Hence T has at least one fixed point on $(\Omega_R \setminus \overline{\Omega}_r) \cap (P\times P)$. Thus (1.1) has at least one positive solution $(u,v) = (L^{m-1}w, L^{n-1}z)$. This completes the proof. □

4. Examples

In this section we provide two examples to illustrate the applicability of our main results. In what follows we assume that α_{ij}, β_{ik} $(i = 1, 2, j = 1, \ldots, m, k = 1, \ldots, n)$ are all nonnegative constants.

Example 1. Let $f_i(t,x,y) := \left(\sum_{j=1}^m \alpha_{ij}x_i + \sum_{k=1}^n \beta_{ik}y_j\right)^{\mu_i}$ where $\mu_i > 1$ $(i = 1, 2)$. There are two cases to be considered.

Case 1. $\sum_{j=1}^m \alpha_{1j} > 0, \sum_{j=1}^n \beta_{2k} > 0$. Take a positive number ω so that

$$\omega > \max\left\{1/\sum_{j=1}^m \alpha_{1j}(r(L))^{m-j+1}, 1/\sum_{k=1}^n \beta_{2k}(r(L))^{n-k+1}\right\}.$$

Let $a_{1j} := \omega\alpha_{1j}, a_{2j} := 0$ $(j = 1, \ldots, m)$, $b_{1k} := 0, b_{2k} := \omega\beta_{2k}$ $(k = 1, \ldots, n)$. Then $a_1 = \sum_{j=1}^m a_{1j}(r(L))^{m-j+1} > 1$, $a_2 = b_1 = 0, b_2 = \sum_{k=1}^n b_{2k}(r(L))^{n-k+1} > 1$. Thus $A_1 = \begin{pmatrix} a_1 - 1 & b_1 \\ a_2 & b_2 - 1\end{pmatrix}$ is invertible with A_1^{-1} nonnegative (see Remark 2). In addition, there is $c > 0$ such that the inequalities in (H3) hold. This means (H3) is satisfied in this case.

Case 2. $\sum_{j=1}^m \alpha_{2j} > 0, \sum_{j=1}^n \beta_{1k} > 0$. Take a positive number ω so that

$$\omega^2 \sum_{j=1}^m \alpha_{2j}(r(L))^{m-j+1} \cdot \sum_{k=1}^n \beta_{1k}(r(L))^{n-k+1} > 1.$$

Let $a_{2j} := \omega\alpha_{2j}, a_{1j} := 0$ $(j = 1, \ldots, m)$, $b_{1k} := \omega\beta_{2k}, b_{2k} := 0$ $(k = 1, \ldots, n)$. Then $a_2b_1 = \sum_{j=1}^m a_{2j}(r(L))^{m-j+1} \cdot \sum_{k=1}^n b_{1k}(r(L))^{n-k+1} > 1$, $a_1 = b_2 = 0$. Thus $A_1 = \begin{pmatrix} a_1 - 1 & b_1 \\ a_2 & b_2 - 1\end{pmatrix}$ is invertible with A_1^{-1} nonnegative (see Remark 2). In addition, there is $c > 0$ such that the inequalities in (H3) hold. This means (H3) is satisfied in this case.

It is easy to verify that (H4) is satisfied in both cases listed above.

Example 2. Let $f_i(t,x,y) := \left(\sum_{j=1}^m \alpha_{ij}x_i + \sum_{k=1}^n \beta_{ik}y_j\right)^{\mu_i}$ where $0 < \mu_i < 1$ $(i = 1, 2)$. Then there are also two cases to be considered.

Case 1. $\sum_{j=1}^m \alpha_{1j} > 0, \sum_{j=1}^n \beta_{2k} > 0$.

Case 2. $\sum_{j=1}^n \beta_{1k} > 0, \sum_{j=1}^m \alpha_{2j} > 0$.

It is easy to verify that (H5) and (H6) are all satisfied in both cases listed above.

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